# Toric Degenerations of Bézier Patches

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The control polygon of a rational Bézier curve is well-defined and has geometric significance; there is a sequence of weights under which the limiting position of the curve is the control polygon. For a rational Bézier surface patch, there are many possible polyhedral control structures, and none is canonical. We propose a not necessarily polyhedral control structure for rational surface patches, regular control surfaces, which are certain $C^0$ spline surfaces. While not unique, regular control surfaces are exactly the possible limiting positions of a rational Bézier patch when the weights vary.

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## 1. INTRODUCTION

In geometric modeling of curves and surfaces, the overall shape of an individual patch is intuitively governed by the placement of control points, and a rational patch may be finely tuned by altering the weights of the basis functions; large weights pull the patch towards the corresponding control points. The control points also have a global meaning as the patch lies within the convex hull of the control points, for any choice of weights.

This convex hull may be indicated by drawing some edges between the control points. The rational bicubic tensor product patches in Figure 1 have the same weights but different control points, and the same $3 \times 3$ grid of edges drawn between the control points. Unlike the control points or their convex hulls, there is no canonical choice of these edges. We paraphrase a question posed to us by Carl de Boor and Ron Goldman: What is the significance for modeling of such control structures (control points plus edges)?

We provide an answer to this question. These control structures, the triangles, quadrilaterals, and other shapes implied by these edges, encode limiting positions of the patch when the weights assume extreme values. Our main results are that the only possible limiting positions of a patch are the control structures arising from regular decompositions (see Section 4) of the points indexing its basis functions and control points, and any such regular control structure is the limiting position of some sequence of patches. Figure 2 shows rational bicubic patches with the control points of Figure 1 and extreme weights. Each is very close to a composite of nine bilinear tensor product patches, corresponding to the nine quadrilaterals in their control structures. The control points of each limiting bilinear patch are the corners of the corresponding quadrilateral. These limiting bilinear patches are all planar on the left, while only the corner quadrilaterals are planar on the right.

The control structure in these examples, which is superimposed on the patch, is a regular decomposition of the $3 \times 3$ grid underlying a bicubic patch. It is regular as it is induced from the upper convex hull of the graph of a function on the 16 grid points. Such a function could be 0 at the four corners, 2 at the four interior points, and 1 at the remaining eight edge points. Figure 3 shows this decomposition on the left together with an irregular decomposition on the right. (If the second decomposition were the upper convex hull of the graph of a function on the grid points, and we assume—as we may—that the central square is flat, then the value of the function at a vertex is lower than the values at a clockwise neighbor, which is impossible outside of Escher woodcuts.)

Such control structures and limiting patches were considered in Craciun et al. [2010], but were restricted to triangulations; this restricted the scope of the results. Our results hold in complete generality and like those of Craciun et al. [2010], rely upon a

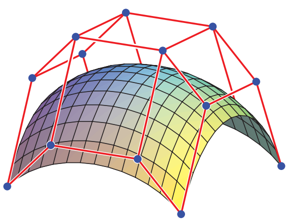


Fig. 1. Two rational bicubic patches.

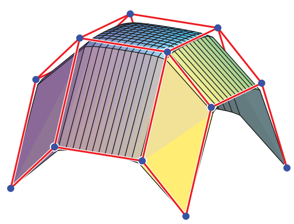


Fig. 2. Two rational bicubic patches with extreme weights.

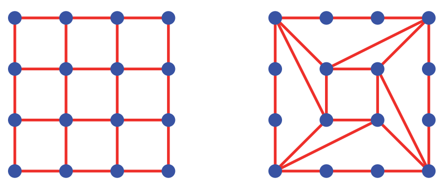


Fig. 3. Regular and irregular decompositions.

construction in computational algebraic geometry called a toric degeneration [Gel′fand et al. 1994, Chapter 8.3.1].

While our primary interest is to explain the meaning of control nets for the classical rational tensor product patches and rational Bézier triangles, we work in the generality of Krasauskas' toric Bézier patches [Krasauskas 2002, 2006]. This is because any polygon may arise in a regular decomposition of the points underlying a classical patch. Figure 4 shows a regular decomposition of the points in the $2 \times 2$ grid underlying a biquadratic patch and on the right is a degenerate patch, which consists of four triangles and Krasauskas's double pillow. The pillow corresponds to the central quadrilateral in the $2 \times 2$ grid, with the "free" internal control point corresponding to the center point of the grid.

Our definitions and arguments make sense in any dimension. The body of this article treats surface patches, but the proofs in the Appendix will be given for patches of any dimension.

We do not address the variation diminishing property, which is another fundamental global aspect of the control polygon of a rational Bézier curve. This states that the number of points in which a Bézier curve meets a line is bounded by number of points in which its control polygon meets the same line. Generalizing this to surfaces is important and interesting, but we currently do not know how to formulate variation diminishing for general surface patches. We remark that this is similar to the open problem of finding a satisfactory multivariate generalization of Descartes' rule of signs.

We first recall basics of rational Bézier triangles and rational tensor product patches and their control nets. Next, we present Krasauskas' toric Bézier patches and introduce the crucial notion of a regular polyhedral decomposition. In the last section we define the main object in this article, a regular control surface, which is a union of toric Bézier patches governed by a regular decomposition.

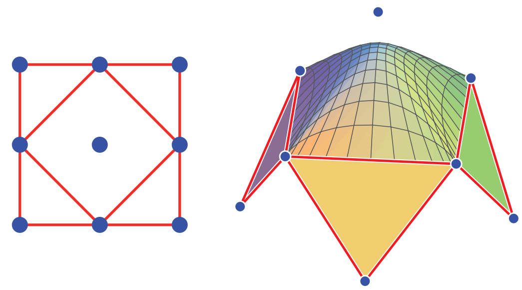


Fig. 4. Degenerate biquadratic patch containing a pillow.

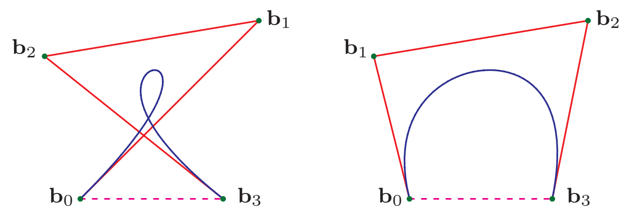



Fig. 5. Rational cubic Bézier planar curves with their control polygons.

We also state our main theorems, Theorem 1, that regular control surfaces are limits of toric Bézier patches, and Theorem 2, that if a patch is sufficiently close to a control surface, then that control surface must be regular. Proofs appear in the Appendix, where we work in the generality of toric patches in arbitrary dimension. Our main tools are results of Kapranov et al. [1991, 1992] which identify all possible toric degenerations of a projective toric variety.

## 2. BÉZIER PATCHES AND CONTROL NETS

We define rational Bézier curves and surfaces and tensor product patches in a form that is convenient for our discussion, and then describe their control nets. Our definition differs from the standard formulation [Farin 1997] in that different domains are used for different degrees. Write $\mathbb{R}_{\geq}$ for the nonnegative real numbers and $\mathbb{R}_{>}$ for the positive real numbers.

Let $d$ be a positive integer. For each $i = 0, \dots, d$ define the *Bernstein polynomial* $\beta_{i;d}(x)$,

$$\beta_{i;d}(x) := x^i(d-x)^{d-i} .$$

(Substituting $x = dy$ and multiplying by $\binom{d}{i} d^{-d}$ for normalization, this becomes the usual Bernstein polynomial. We omit the binomial coefficients, for it is these unadorned Bernstein polynomials which the toric basis functions of Section 3 generalize.) Given weights $w_0, \dots, w_d \in \mathbb{R}_{>}$ and control points $\mathbf{b}_0, \dots, \mathbf{b}_d \in \mathbb{R}^n$ $(n = 2, 3)$, we have the parameterized *rational Bézier curve*

$$F(x) := \frac{\sum_{i=0}^{d} w_i \mathbf{b}_i \beta_{i;d}(x)}{\sum_{i=0}^{d} w_i \beta_{i;d}(x)} : [0, d] \longrightarrow \mathbb{R}^n .$$

Our domain is $[0, d]$ rather than $[0, 1]$, for this is the convention for toric patches.

The *control polygon* of the curve is the union of segments $\overline{\mathbf{b}_0, \mathbf{b}_1}, \dots, \overline{\mathbf{b}_{d-1}, \mathbf{b}_d}$. Figure 5 shows two rational cubic Bézier planar curves with their control polygons. There are two standard ways to extend this to surfaces. The most straightforward gives rational tensor product patches. Let $c, d$ be positive integers and for each $i = 0, \dots, c$ and $j = 0, \dots, d$ let $w_{(i,j)} \in \mathbb{R}_{>}$ and $\mathbf{b}_{(i,j)} \in \mathbb{R}^3$ be a weight and a control point. The associated rational tensor product patch of bidegree $(c, d)$ is the image of the map $[0, c]\times[0, d] \to \mathbb{R}^3$,

$$F(x, y) \;:=\; \frac{\sum_{i=0}^{c}\sum_{j=0}^{d} w_{(i,j)}\mathbf{b}_{(i,j)}\beta_{i;c}(x)\beta_{j;d}(y)}{\sum_{i=0}^{c}\sum_{j=0}^{d} w_{(i,j)}\beta_{i;c}(x)\beta_{j;d}(y)}\,.$$

Triangular Bézier patches are another extension. Set

$$d\triangle \;:=\; \{(x, y) \in \mathbb{R}^2 \mid 0 \le x, y \text{ and } x + y \le d\}$$

and set $\mathcal{A} := d\triangle \cap \mathbb{Z}^2$, the points with integer coordinates (lattice points) in the triangle $d\triangle$. For $(i, j) \in \mathcal{A}$, we have the bivariate Bernstein polynomial

$$\beta_{(i,j);d}(x, y) \;:=\; x^i y^j (d - x - y)^{d-i-j}\,.$$

Given weights $w = \{w_{(i,j)} \mid (i, j) \in \mathcal{A}\}$ and control points $\mathcal{B} = \{\mathbf{b}_{(i,j)} \mid (i, j) \in \mathcal{A}\}$, the associated triangular rational Bézier patch is the image of the map $d\triangle \to \mathbb{R}^3$,

$$F(x, y) \;:=\; \frac{\sum_{(i,j)\in\mathcal{A}} w_{(i,j)}\mathbf{b}_{(i,j)}\beta_{(i,j);d}(x, y)}{\sum_{(i,j)\in\mathcal{A}} w_{(i,j)}\beta_{(i,j);d}(x, y)}\,.$$

The control points of a Bézier curve are connected in sequence to give the control polygon, which is a piecewise linear caricature of the curve. For a surface patch there are, however, many ways to interpolate the control points by edges to form a control net. There also may not be a way to fill in these edges with polygons to form a control polytope. Even when this is possible, the significance of this structure for the shape of the patch is not evident, except in special cases. For example, Chang and Davis [1984] show for triangular Bézier patches that if the control points are the graph of a convex function over the lattice points, *and this induces a particular triangulation called the Bézier net*, then the patch is convex.

## 3. TORIC PATCHES AND TORIC VARIETIES

Krasauskas's toric patches [Krasauskas 2002] are a natural extension of rational Bézier triangles and rational tensor product patches to arbitrary polygons whose vertices have integer coordinates, called *lattice polygons*. They are based on toric varieties [Cox et al. 2011; Fulton 1993] from algebraic geometry which get their name as they are natural compactifications of algebraic tori $(\mathbb{C}^*)^n$, where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. They are naturally associated to lattice polygons (and in higher dimensions, lattice polytopes), and the positive real part [Fulton 1993, Chapter 4; Sottile 2003] of a toric variety is canonically identified with the corresponding polygon/polytope.

We simplify our notation, writing $x = (x_1, x_2)$ for points of $\mathbb{R}^2$. Toric patches begin with a finite set $\mathcal{A} \subset \mathbb{Z}^2$ of (integer) lattice points. The convex hull of $\mathcal{A}$ is the set of all convex combinations

$$\sum_{\mathbf{a}\in\mathcal{A}} p_{\mathbf{a}}\mathbf{a} \qquad \text{where} \qquad p_{\mathbf{a}} \ge 0 \qquad \text{and} \qquad 1 \;=\; \sum_{\mathbf{a}\in\mathcal{A}} p_{\mathbf{a}}$$

of points of $\mathcal{A}$, which is a lattice polygon and is written $\Delta_{\mathcal{A}}$. To each edge $e$ of $\Delta_{\mathcal{A}}$, there is a valid inequality $h_e(x) \ge 0$ on $\Delta_{\mathcal{A}}$, where $h_e(x)$ is a linear polynomial with relatively prime integer coefficients that vanishes on the edge $e$. For example, if $\mathcal{A} = d\triangle \cap \mathbb{Z}^2$ and $\Delta_{\mathcal{A}} = d\triangle$, then the inequalities are

$$x_1 \ge 0, \quad x_2 \ge 0, \quad \text{and} \quad d - x_1 - x_2 \ge 0\,,$$

and the central quadrilateral of Figure 4 has inequalities

$$x_1 + x_2 - 1,\;\; 1 + x_1 - x_2,\;\; 3 - x_1 - x_2,\;\; 1 + x_2 - x_1 \;\ge\; 0.$$

Let $E$ be the set of edges of the polygon $\Delta_{\mathcal{A}}$. To each lattice point $\mathbf{a} \in \mathcal{A}$, define the *toric basis function* $\beta_{\mathbf{a},\mathcal{A}} \colon \Delta_{\mathcal{A}} \to \mathbb{R}$ to be

$$\beta_{\mathbf{a},\mathcal{A}}(x) \;:=\; \prod_{e\in E} h_e(x)^{h_e(\mathbf{a})}.$$

This is strictly positive in the interior of $\Delta_{\mathcal{A}}$. If $\mathbf{a}$ lies on an edge $e$ of $\Delta_{\mathcal{A}}$, then $\beta_{\mathbf{a},\mathcal{A}}$ is strictly positive on the relative interior of $e$, and if $\mathbf{a}$ is a vertex, then $\beta_{\mathbf{a},\mathcal{A}}(\mathbf{a}) > 0$. In particular the toric basis functions have no common zeroes in $\Delta_{\mathcal{A}}$.

Observe that the toric basis functions for $\mathcal{A} = [0, c] \times [0, d] \cap \mathbb{Z}^2$ and $\mathcal{A} = d\triangle \cap \mathbb{Z}^2$ are equal to the Bernstein polynomials $\beta_{i;c}(x_1)\beta_{j;d}(x_2)$ and $\beta_{(i,j);d}(x_1, x_2)$ underlying the tensor product and triangular Bézier patches.

Toric patches also require *weights* and control points. Let $\#\mathcal{A}$ be the number of points in $\mathcal{A}$. Let $\mathbb{R}^{\mathcal{A}}_{>}$ be $\mathbb{R}^{\#\mathcal{A}}_{>}$ with coordinates $(z_{\mathbf{a}} \in \mathbb{R}_{>} \mid \mathbf{a} \in \mathcal{A})$ indexed by elements of $\mathcal{A}$. A toric Bézier patch of shape $\mathcal{A}$ is given by a collection of positive weights $w = (w_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}) \in \mathbb{R}^{\mathcal{A}}_{>}$ and control points $\mathcal{B} = \{\mathbf{b}_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^3$. These are used to define a map $\Delta_{\mathcal{A}} \to \mathbb{R}^3$,

$$F_{\mathcal{A},w,\mathcal{B}}(x) \;:=\; \frac{\sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}}\mathbf{b}_{\mathbf{a}}\beta_{\mathbf{a},\mathcal{A}}(x)}{\sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}}\beta_{\mathbf{a},\mathcal{A}}(x)}\,. \tag{1}$$

Since the toric basis functions are nonnegative on $\Delta_{\mathcal{A}}$ and have no common zeroes, this denominator is strictly positive on $\Delta_{\mathcal{A}}$. Write $Y_{\mathcal{A},w,\mathcal{B}}$ for the image of $\Delta_{\mathcal{A}}$ under the map $F_{\mathcal{A},w,\mathcal{B}}$, which is a *toric Bézier patch* of shape $\mathcal{A}$.

We will show that the map $F_{\mathcal{A},w,\mathcal{B}} \colon \Delta_{\mathcal{A}} \to \mathbb{R}^3$ factors as

$$F_{\mathcal{A},w,\mathcal{B}} \,:\; \Delta_{\mathcal{A}} \xrightarrow{\;\beta_{\mathcal{A}}\;} \triangle^{\mathcal{A}} \xrightarrow{\;w\cdot\;} \triangle^{\mathcal{A}} \xrightarrow{\;\pi_{\mathcal{B}}\;} \mathbb{R}^3\,, \tag{2}$$

where $\triangle^{\mathcal{A}} \subset \mathbb{R}^{\mathcal{A}}$ is the standard simplex of dimension $\#\mathcal{A}-1$, which we identify with the nonnegative orthant modulo $\mathbb{R}_{>}$, the map $\beta_{\mathcal{A}}$ is induced by the toric basis functions $\beta_{\mathbf{a},\mathcal{A}}$, the map $w\cdot$ is induced by coordinatewise multiplication by the weights $w$, and the map $\pi_{\mathcal{B}}$ is a projection given by the control points $\mathcal{B}$. The purpose of this factorization is to clarify the role of the weights in a toric patch by isolating their effect. The image $\beta_{\mathcal{A}}(\Delta_{\mathcal{A}}) \subset \triangle^{\mathcal{A}}$ is a standard toric variety $X_{\mathcal{A}}$. Acting on this by the map $w\cdot$ gives a translated toric variety $X_{\mathcal{A},w}$, which we call a *lift* of the patch $Y_{\mathcal{A},w,\mathcal{B}}$ as its image under the projection $\pi_{\mathcal{B}}$ is $Y_{\mathcal{A},w,\mathcal{B}}$. We use results on the limiting position of the translates $X_{\mathcal{A},w}$ as the weights are allowed to vary, which are called toric degenerations.

We make this precise. Let $\mathbb{R}^{\mathcal{A}}_{\ge}$ be $\mathbb{R}^{\#\mathcal{A}}_{\ge}$ with coordinates $(z_{\mathbf{a}} \in \mathbb{R}_{\ge} \mid \mathbf{a} \in \mathcal{A})$ indexed by elements of $\mathcal{A}$. The standard simplex

$$\triangle^{\mathcal{A}} \;:=\; \Big\{ z \in \mathbb{R}^{\mathcal{A}}_{\ge} \mid \sum_{\mathbf{a}\in\mathcal{A}} z_{\mathbf{a}} = 1 \Big\}$$

is the convex hull of the standard basis in $\mathbb{R}^{\mathcal{A}}$, and so has natural barycentric coordinates. It is also the quotient of the nonnegative orthant under multiplication by positive scalars, which gives it natural homogeneous coordinates, in which we identify $[z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}]$ with $[t \cdot z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}]$ when $z_{\mathbf{a}} \ge 0$ and $t > 0$. These homogeneous coordinates correspond to barycentric coordinates as follows.

$$[z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}] \iff \frac{1}{\sum_{\mathbf{a}\in\mathcal{A}} z_{\mathbf{a}}}(z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}) \tag{3}$$

Geometrically, $[z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}] \in \triangle^{\mathcal{A}}$ is the unique point where the ray $\mathbb{R}_{>} \cdot (z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A})$ meets the simplex $\triangle^{\mathcal{A}}$.

Let $\beta_{\mathcal{A}} \colon \Delta_{\mathcal{A}} \to \triangle^{\mathcal{A}}$ be the map $\beta_{\mathcal{A}}(x) = [\beta_{\mathbf{a},\mathcal{A}}(x) \mid \mathbf{a} \in \mathcal{A}]$. A vector of weights $w \in \mathbb{R}^{\mathcal{A}}_{>}$ defines a map $w\cdot \colon \triangle^{\mathcal{A}} \to \triangle^{\mathcal{A}}$,

$$w \cdot [z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}] \;=\; [w_{\mathbf{a}} z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}]\,.$$

Given control points $\mathcal{B}$, define the linear map $\pi_{\mathcal{B}} \colon \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^3$ by

$$\pi_{\mathcal{B}}(z) \;:=\; \sum_{\mathbf{a}\in\mathcal{A}} \mathbf{b}_{\mathbf{a}} z_{\mathbf{a}}\,.$$

The image of the simplex $\triangle^{\mathcal{A}}$ under $\pi_{\mathcal{B}}$ is the convex hull of the control points $\mathcal{B}$, and by these definitions, the map $F_{\mathcal{A},w,\mathcal{B}}$ in (1) defining the toric Bézier patch is the composition (2).

We call $Y_{\mathcal{A},w,\mathcal{B}}$ a toric patch because the image $\beta_{\mathcal{A}}(\Delta_{\mathcal{A}})$ is a toric variety. Elements $\mathbf{a}$ of $\mathbb{Z}^2$ are exponents of monomials,

$$\mathbf{a} \;=\; (a_1, a_2) \;\longleftrightarrow\; x_1^{a_1} x_2^{a_2}\,,$$

which we will write as $x^{\mathbf{a}}$. The points of $\mathcal{A}$ define a map $\varphi_{\mathcal{A}} \colon \mathbb{R}^2_{>} \to \triangle^{\mathcal{A}}$ by

$$\varphi_{\mathcal{A}}(x) \;:=\; [x^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}]\,.$$

The closure in $\triangle^{\mathcal{A}}$ of the image of $\varphi_{\mathcal{A}}$ is the *toric variety* $X_{\mathcal{A}}$. We have the following result of Krasauskas [2002].

PROPOSITION 1 [KRASAUSKAS 2002]. *The image of $\Delta_{\mathcal{A}}$ under the map $\beta_{\mathcal{A}}$ is the toric variety $X_{\mathcal{A}}$.*

Toric patches share with rational Bézier patches the following recursive structure. If $\mathbf{a}$ is a vertex of $\Delta_{\mathcal{A}}$, then $\mathbf{b}_{\mathbf{a}} = F_{\mathcal{A},w,\mathcal{B}}(\mathbf{a})$ is a point in the patch. If $e$ is the edge between two vertices of $\Delta_{\mathcal{A}}$, then the restriction $F_{\mathcal{A},w,\mathcal{B}}|_e$ of $F_{\mathcal{A},w,\mathcal{B}}$ to $e$ is the one-dimensional toric patch given by the points of $\mathcal{A}$ lying on $e$ and the corresponding weights, which is a rational Bézier curve. For example, the edges of the patches in Figure 1 are all rational cubic Bézier curves.

## 4. REGULAR POLYHEDRAL DECOMPOSITIONS

We recall the definitions of regular (or coherent) polyhedral subdivisions from geometric combinatorics, which were introduced in Gel′fand et al. [1994, Section 7.2]. Because subdivision has a different meaning in modeling, we instead use the term *decomposition*. Let $\mathcal{A} \subset \mathbb{R}^2$ be a finite set and suppose that $\lambda \colon \mathcal{A} \to \mathbb{R}$ is a function. We use $\lambda$ to lift the points of $\mathcal{A}$ into $\mathbb{R}^3$. Let $P_\lambda$ be the convex hull of the lifted points,

$$P_\lambda = \mathrm{conv}\{(\mathbf{a}, \lambda(\mathbf{a})) \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^3.$$

Each face of $P_\lambda$ has an outward pointing normal vector, and its *upper facets* are those whose normal has positive last coordinate. If we project these upper facets back to $\mathbb{R}^2$, they cover the polygon $\Delta_{\mathcal{A}}$ and are the facets of the *regular polyhedral decomposition* $\mathcal{T}_\lambda$ of $\Delta_{\mathcal{A}}$ induced by $\lambda$. (Taking lower facets gives $\mathcal{T}_{-\lambda}$, so it is no loss of generality to work with upper facets.)

The edges and vertices of $\mathcal{T}_\lambda$ are the images of the edges and vertices lying on upper facets. Figure 6 shows the upper facets and the regular polyhedral decompositions given by two different lifting functions for the points $\mathcal{A}$ underlying a biquadratic tensor product patch. More generally, a *polyhedral decomposition* of $\Delta_{\mathcal{A}}$ is a collection $\mathcal{T}$ of polygons, line segments, and points of $\mathcal{A}$, whose union is $\Delta_{\mathcal{A}}$, where any edge, vertex, or endpoint of a segment also lies in $\mathcal{T}$, and any two elements of $\mathcal{T}$ are either disjoint or their intersection is an element of $\mathcal{T}$. A decomposition $\mathcal{T}$ is *regular* if it is induced from a lifting function.

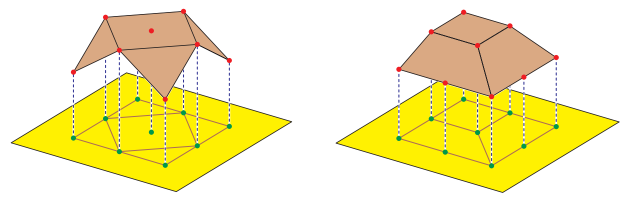


Fig. 6. Two upper hulls and decompositions for biquadratic patches.

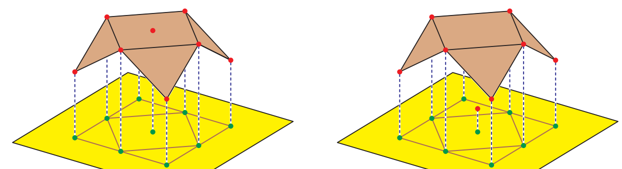


Fig. 7. Two different decompositions for biquadratic patches.

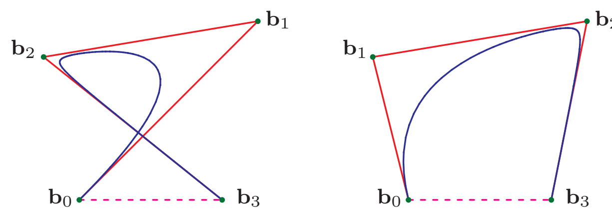



Fig. 8. Rational cubic Bézier planar curves with $t = 5$.

A *decomposition* $\mathcal{S}$ of the configuration $\mathcal{A}$ of points is a collection $\mathcal{S}$ of subsets of $\mathcal{A}$ called *faces*. The convex hulls of these faces are required to be the polygons, line segments, and vertices of a polyhedral decomposition $\mathcal{T}(\mathcal{S})$ of $\Delta_{\mathcal{A}}$. In particular, the intersection of any face with the convex hull $\Delta_{\mathcal{F}}$ of another face $\mathcal{F}$ of $\mathcal{S}$ is either empty, a vertex of $\Delta_{\mathcal{F}}$, or the points of $\mathcal{F}$ lying in some edge of $\Delta_{\mathcal{F}}$. A face $\mathcal{F}$ is a *facet*, *edge*, or *vertex* of $\mathcal{S}$ as its convex hull $\Delta_{\mathcal{F}}$ is a polygon, line segment, or vertex. The decomposition $\mathcal{S}$ is *regular* if the polyhedral decomposition $\mathcal{T}(\mathcal{S})$ is regular. We remark that not every point of $\mathcal{A}$ need lie in some face of a decomposition.

Figure 7 shows two different lifting functions that induce the same regular polyhedral decomposition of the $2 \times 2$ square underlying a biquadratic patch, but different regular decompositions of $\mathcal{A}$. The center point of $\mathcal{A}$ does not lie in any face of the decomposition on the right as its lift does not lie on any upper facet.

Here is a one-dimensional example. Let $\lambda$ take the values $\{0, 1, 2, 0\}$ on the points $\{0, 1, 2, 3\}$ underlying rational cubic Bézier curves. This induces a regular decomposition of $\{0, 1, 2, 3\}$ with facets

$$\{0, 1, 2\} \qquad \text{and} \qquad \{2, 3\}\,. \tag{4}$$

## 5. REGULAR CONTROL SURFACES

Regular control surfaces are possible limiting positions of patches. We first illustrate these notions on a rational cubic curve in the plane. The curves of Figure 5 have weights $(1, 4, 4, 1)$ at the points $0\,,\ 1\,,\ 2\,,\ 3$, respectively. We use the lifting function inducing the decomposition (4) to define a family of weights $(1 \cdot t^0, 4 \cdot t^1, 4 \cdot t^2, 1 \cdot t^0) = (1, 4t, 4t^2, 1)$ for $t \in \mathbb{R}_{>}$. Figure 8 shows the curves with $t = 5$ and the control points of Figure 5.

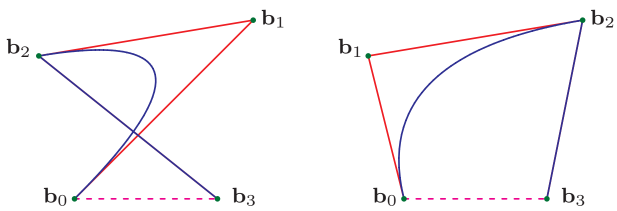



Fig. 9. Regular control curves.

To consider the limit as $t \to \infty$, write the Bernstein polynomials in homogeneous form as $\beta_{i;3} := u^i v^{3-i}$ for $i = 0, \ldots, 3$, for then the cubic curve is the image of points $(u, v) \in (\mathbb{R}_>)^2$.

Limiting positions are given by restrictions to the facets of the decomposition (4). Multiplying the $\beta_{i;3}$ by the weights and restricting to each facet, we get basis functions

$$\{v^3, 4tv^2u, 4t^2vu^2\}, \qquad \text{and} \qquad \{4t^2vu^2, u^3\}.$$

These give rational Bézier curves

$$\frac{v^3\mathbf{b}_0 + 4tv^2u\mathbf{b}_1 + 4t^2vu^2\mathbf{b}_2}{v^3 + 4tv^2u + 4t^2vu^2} \qquad \text{and} \qquad \frac{4t^2vu^2\mathbf{b}_2 + u^3\mathbf{b}_3}{4t^2vu^2 + u^3}.$$

Dividing out the common factor of $v$ from the first and replacing $tu$ by $u$, and similarly dividing out $u^2$ from the second and replacing $vt^2$ by $v$, we get

$$\frac{v^2\mathbf{b}_0 + 4vu\mathbf{b}_1 + 4u^2\mathbf{b}_2}{v^2 + 4vu + 4u^2} \qquad \text{and} \qquad \frac{4v\mathbf{b}_2 + u\mathbf{b}_3}{4v + u},$$

which are rational quadratic and linear Bézier curves. Figure 9 shows these curves with the control points of Figure 8. These are regular control curves induced by the decomposition (4).

This restriction to facets followed by a monomial reparametrization allowed the determination of the limiting position of the curve as $t \to \infty$. While a sequence of such restrictions and reparametrizations leads to general control curves, these operations are not sufficient for surfaces.

We describe the possible limiting positions of toric surface patches. Let $\mathcal{A} \subset \mathbb{Z}^2$ be a finite set, $w \in \mathbb{R}^{\mathcal{A}}_>$ be a vector of weights, and $\mathcal{B} = \{\mathbf{b}_\mathbf{a} \mid \mathbf{a} \in \mathcal{A}\}$ be control points for a toric patch $Y_{\mathcal{A},w,\mathcal{B}}$ of shape $\mathcal{A}$.

Suppose that we have a decomposition $\mathcal{S}$ of $\mathcal{A}$. We may use the weights $w$ and control points $\mathcal{B}$ indexed by elements of a facet $\mathcal{F}$ as weights and control points for a toric patch of shape $\mathcal{F}$, written $Y_{\mathcal{F},w|_\mathcal{F},\mathcal{B}|_\mathcal{F}}$. In fact, this can be done for any face of $\mathcal{S}$. The union

$$Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}) := \bigcup_{\mathcal{F}\in\mathcal{S}} Y_{\mathcal{F},w|_\mathcal{F},\mathcal{B}|_\mathcal{F}},$$

of these patches is the *control surface* induced by the decomposition $\mathcal{S}$. As the domain of a patch of shape $\mathcal{F}$ is the convex hull $\Delta_\mathcal{F}$ of $\mathcal{F}$ and faces of toric patches are again toric patches, the control surface of a patch induced by a decomposition is naturally a $C^0$ spline surface. A control surface $Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S})$ is *regular* if the decomposition $\mathcal{S}$ is regular.

Figure 10 shows the control surfaces of the bicubic patches from Figure 1. These control surfaces are regular as they are induced by the $3 \times 3$ grid, which is a regular decomposition. We invite the reader to compare them to the patches of Figure 2. Figure 11 shows the irregular decomposition of the $3 \times 3$ grid from Figure 3 and a corresponding irregular control surface. The central quadrilateral $A$ in the decomposition corresponds to the bilinear patch at the top, the triangle $B$ in the decomposition corresponds to the indicated flat

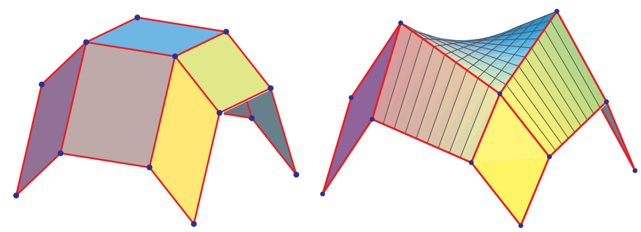


Fig. 10. Regular control surfaces.

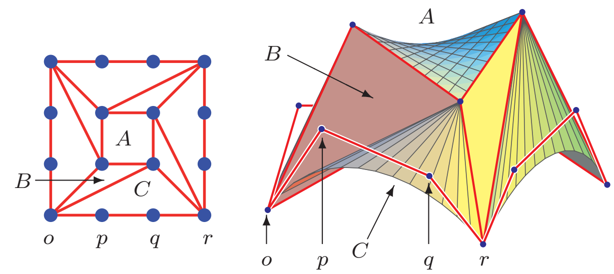



Fig. 11. Irregular decomposition and an irregular control surface.

triangle, and the triangle $C$ with points $o, p, q, r$ along one edge corresponds to the singular ruled cubic in the surface. The polygonal frame formed by the corresponding control points on the right is the control polygon for this edge of $C$, which is a rational cubic Bézier curve.

We show that regular control surfaces are exactly the possible limits of toric patches when the control points $\mathcal{B}$ are fixed but the weights $w$ are allowed to vary. In particular, the irregular control surface Figure 11 cannot be the limit of toric Bézier patches.

Let $\lambda\colon \mathcal{A} \to \mathbb{R}$ be a lifting function. We use this and a given set of weights $w = \{w_\mathbf{a} \in \mathbb{R}_> \mid \mathbf{a} \in \mathcal{A}\}$ to get a set of weights which depends upon a parameter, $w_\lambda(t) := \{t^{\lambda(\mathbf{a})}w_\mathbf{a} \mid \mathbf{a} \in \mathcal{A}\}$. These weights are used to define a *toric degeneration* of the patch,

$$F_{\mathcal{A},w,\mathcal{B},\lambda}(x;t) := \frac{\sum_{\mathbf{a}\in\mathcal{A}} t^{\lambda(\mathbf{a})} w_\mathbf{a} \mathbf{b}_\mathbf{a} \beta_\mathbf{a}(x)}{\sum_{\mathbf{a}\in\mathcal{A}} t^{\lambda(\mathbf{a})} w_\mathbf{a} \beta_\mathbf{a}(x)}.$$

Let $\mathcal{S}_\lambda$ be the regular decomposition of $\mathcal{A}$ induced by $\lambda$. We show that the regular control surface $Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}_\lambda)$ induced by $\mathcal{S}_\lambda$ is the limit of the patches $Y_{\mathcal{A},w,\mathcal{B},\lambda}(t)$ parameterized by $F_{\mathcal{A},w,\mathcal{B},\lambda}(x;t)$ as $t \to \infty$. We distinguish between the parametrization $F_{\mathcal{A},w,\mathcal{B},\lambda}(x;t)$ and its image the patch $Y_{\mathcal{A},w,\mathcal{B},\lambda}(t)$, not only because they are distinct objects, but because there is no limiting parametrization, despite there being a well-defined limiting position of patches.

This limit is with respect to the Hausdorff distance between two subsets of $\mathbb{R}^3$. Two subsets $X$ and $Y$ of $\mathbb{R}^3$ are *within Hausdorff distance* $\epsilon$ if for every point $x$ of $X$ there is some point $y$ of $Y$ within a distance $\epsilon$ of $x$, and vice versa. With this notion of distance, we have the following result.

THEOREM 1. $\lim_{t\to\infty} Y_{\mathcal{A},w,\mathcal{B},\lambda}(t) = Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}_\lambda)$.

That is, for every $\epsilon > 0$ there is a number $M$ such that if $t \geq M$, then the patch $Y_{\mathcal{A},w,\mathcal{B},\lambda}(t)$ and the regular control surface $Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}_\lambda)$ are within Hausdorff distance $\epsilon$.

We illustrate Theorem 1. On the left in the following graphic are the weights of a bicubic patch, in the center are the values of a

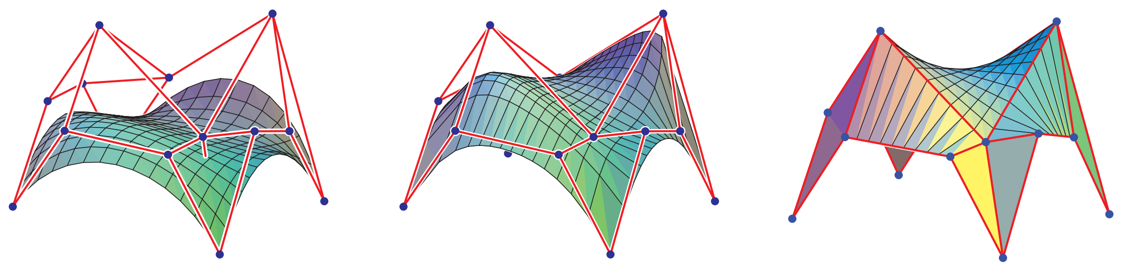
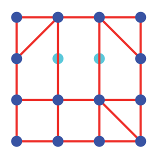


Fig. 12. Toric degeneration of a rational tensor product patch of bidegree (3, 3).

lifting function, and the corresponding regular decomposition is on the right.

| | | | |
|---|---|---|---|
| 1 | 3 | 3 | 1 |
| 3 | 9 | 9 | 3 |
| 3 | 9 | 9 | 3 |
| 1 | 3 | 3 | 1 |

| | | | |
|---|---|---|---|
| 0 | 2 | 2 | 0 |
| 1 | 1 | 1 | 1 |
| 1 | 2 | 2 | 1 |
| 0 | 1 | 1 | 0.5 |



The two lighter points, (1, 2) and (2, 2), lie in no face of the decomposition. Figure 12 shows the toric degeneration of this bicubic patch at values $t = 1$ and $t = 6$, and the regular control surface, all with the indicated control points.

We will prove Theorem 1 in Appendix A. The key idea is the factorization (2) of the map $F_{\mathcal{A},w,\mathcal{B},\lambda}(x; t)$ through the simplex $\Delta^{\mathcal{A}}$. This factorization allows us to study the limit in Theorem 1 by considering the effect of the family of weights $w_\lambda(t)$ on the toric variety $X_{\mathcal{A}}$ in $\Delta^{\mathcal{A}}$. Using equations for $X_{\mathcal{A}}$, we can show that the limit as $t \to \infty$ of the translated toric variety $X_{\mathcal{A},w_\lambda(t)}$ is a regular control surface in $\mathbb{R}^{\mathcal{A}}$ whose projection to $\mathbb{R}^3$ is the regular control surface $Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}_\lambda)$.

Figure 13 shows a toric degeneration of a rational cubic Bézier curve, together with the corresponding degeneration of the curve $X_{\mathcal{A},w}$ in the simplex $\Delta^{\mathcal{A}}$. Here, the weights are $w_\lambda(t) = (1, 3t^2, 3t^2, 1)$. That is, the control points $\mathbf{b}_0$ and $\mathbf{b}_3$ have weight 1, while the internal control points $\mathbf{b}_1$ and $\mathbf{b}_2$ have weights $3t^2$.

By Theorem 1, every regular control surface is the limit of the corresponding patch under a toric degeneration. Our second main result is a converse: If a space $Y$ is the limit of patches of shape $\mathcal{A}$ with control points $\mathcal{B}$, but differing weights, then $Y$ is a regular control surface of shape $\mathcal{A}$ and control points $\mathcal{B}$.

THEOREM 2. *Let $\mathcal{A} \subset \mathbb{Z}^2$ be a finite set and $\mathcal{B} = \{\mathbf{b_a} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^3$ a set of control points. If $Y \subset \mathbb{R}^3$ is a set for which there is a sequence $w^1, w^2, \ldots$ of weights so that*

$$\lim_{i\to\infty} Y_{\mathcal{A},w^i,\mathcal{B}} = Y,$$

*then there is a lifting function $\lambda\colon \mathcal{A} \to \mathbb{R}$ and a weight $w \in \mathbb{R}^{\mathcal{A}}_{>}$ such that $Y = Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}_\lambda)$, a regular control surface.*

To prove Theorem 2, we consider the sequence of translated toric varieties $X_{\mathcal{A},w^i} \subset \Delta^{\mathcal{A}}$. We show how Kapranov et al. [1991, 1992] implies that the set of all translated toric varieties is naturally compactified by the set of all regular control surfaces in $\Delta^{\mathcal{A}}$. Thus some subsequence of $\{X_{\mathcal{A},w^i}\}$ converges to a regular control surface in $\Delta^{\mathcal{A}}$, whose image must coincide with $Y$, implying that $Y$ is a regular control surface. This method of proof does not give a simple way to recover a lifting function $\lambda$ or the weight $w$ from the sequence of weights $w^1, w^2, \ldots$.

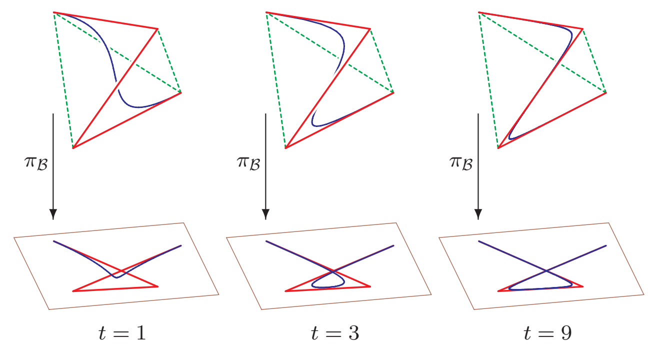



Fig. 13. Toric degenerations of a rational cubic Bézier curve.

We prove Theorem 1 in Appendix A and Theorem 2 in Appendix B. While both require more algebraic geometry than we have assumed so far, Appendix A is more elementary and Appendix B is significantly more sophisticated.

## APPENDIXES

## A. PROOF OF THEOREM 1

Let $d, n$ be positive integers. The definitions and results of Sections 3, 4, and 5, as well as the statements of Theorems 1 and 2 make sense if we replace $\mathcal{A} \subset \mathbb{Z}^2$ by $\mathcal{A} \subset \mathbb{Z}^d$ and $\mathcal{B} \subset \mathbb{R}^3$ by $\mathcal{B} \subset \mathbb{R}^n$. We work here in this generality. This requires straightforward modifications such as replacing polygon by polytope and in general removing restrictions on dimension. We invite the reader to consult Craciun et al. [2010] for a more complete treatment.

If the control points $\mathcal{B}$ are the vertices $\{e_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^{\mathcal{A}}$ of the simplex $\Delta^{\mathcal{A}}$, then the toric patch $Y_{\mathcal{A},w,\mathcal{B}}$ is the (translated) toric variety $X_{\mathcal{A},w}$. Given a decomposition $\mathcal{S}$ of $\mathcal{A}$, write $X_{\mathcal{A},w}(\mathcal{S})$ for the control surface induced by $\mathcal{S}$ when the control points are the vertices of $\Delta^{\mathcal{A}}$. This is the union of patches $X_{\mathcal{F},w|_{\mathcal{F}}}$ over all faces $\mathcal{F}$ of $\mathcal{S}$, and each patch $X_{\mathcal{F},w|_{\mathcal{F}}}$ lies in the face $\Delta^{\mathcal{F}}$ of $\Delta^{\mathcal{A}}$ consisting of points $z$ whose coordinates $z_{\mathbf{a}}$ vanish for $\mathbf{a} \notin \mathcal{F}$.

Then, given any control points $\mathcal{B} \subset \mathbb{R}^n$, we have

$$\pi_{\mathcal{B}}(X_{\mathcal{A},w}) = Y_{\mathcal{A},w,\mathcal{B}} \quad \text{and} \quad \pi_{\mathcal{B}}(X_{\mathcal{A},w}(\mathcal{S})) = Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}).$$

Because of this universality of $X_{\mathcal{A},w}$, $X_{\mathcal{A},w}(\mathcal{S})$, and the map $\pi_{\mathcal{B}}$, it suffices to prove Theorem 1 for limits of the toric variety $X_{\mathcal{A},w}$. Given a function $\lambda\colon \mathcal{A} \to \mathbb{R}$ and a weight $w \in \mathbb{R}^{\mathcal{A}}_{>}$, define the family of weights $w_\lambda(t) = \{t^{\lambda(\mathbf{a})} w_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ for $t \in \mathbb{R}_{>}$.

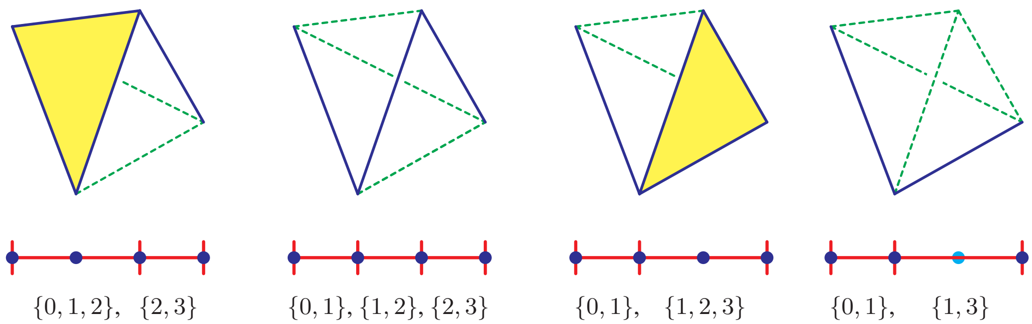



Fig. 14. Geometric realizations for four decompositions.

THEOREM 3. *Let $\lambda\colon \mathcal{A} \to \mathbb{R}$ be a lifting function and $\mathcal{S}_\lambda$ the regular decomposition of $\mathcal{A}$ induced by $\lambda$. Then, for any choice of weights $w \in \mathbb{R}^{\mathcal{A}}_{>}$,*

$$\lim_{t\to\infty} X_{\mathcal{A},w_\lambda(t)} \;=\; X_{\mathcal{A},w}(\mathcal{S}_\lambda)\,.$$

We prove Theorem 3 in two parts. We first show that any accumulation points of $\{X_{\mathcal{A},w_\lambda(t)} \mid t \geq 1\}$ as $t \to \infty$ are contained in the union of the faces $\triangle^{\mathcal{F}}$ of the simplex $\triangle^{\mathcal{A}}$ for each face $\mathcal{F}$ of $\mathcal{S}_\lambda$. Then we show that $X_{\mathcal{F},w|_{\mathcal{F}}}$ is the set of accumulation points contained in the face $\triangle^{\mathcal{F}}$, and in fact each accumulation point is a limit point. This will complete the proof of Theorem 3 as

$$X_{\mathcal{A},w}(\mathcal{S}_\lambda) \;=\; \bigcup_{\mathcal{F}\in\mathcal{S}} X_{\mathcal{F},w|_{\mathcal{F}}}\,.$$

We use homogeneous equations for $X_{\mathcal{A},w}$. Let $\mathbf{1} \in \mathbb{R}^{\mathcal{A}}_{>}$ be the weight with every coordinate 1. Equations for $X_{\mathcal{A},\mathbf{1}}$ were described in Craciun et al. [2010, Proposition B.3] as follows. For every linear relation among the points of $\mathcal{A}$ with nonnegative integer coefficients

$$\sum_{\mathbf{a}\in\mathcal{A}} \alpha_{\mathbf{a}}\mathbf{a} \;=\; \sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}}\mathbf{a} \qquad \text{where} \qquad \sum_{\mathbf{a}\in\mathcal{A}} \alpha_{\mathbf{a}} \;=\; \sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}}\,, \tag{5}$$

with $\alpha_{\mathbf{a}}, \beta_{\mathbf{a}} \in \mathbb{N}$, we have the valid equation for points $z \in X_{\mathcal{A},\mathbf{1}}$,

$$\prod_{\mathbf{a}\in\mathcal{A}} z_{\mathbf{a}}^{\alpha_{\mathbf{a}}} \;=\; \prod_{\mathbf{a}\in\mathcal{A}} z_{\mathbf{a}}^{\beta_{\mathbf{a}}}\,. \tag{6}$$

Conversely, if $z \in \triangle^{\mathcal{A}}$ satisfies Eq. (6) for every relation (5), then $z \in X_{\mathcal{A},\mathbf{1}}$. This follows from the description of toric ideals in Sturmfels [1996, Chapter 4].

Since the toric variety $X_{\mathcal{A},w}$ is obtained from $X_{\mathcal{A},\mathbf{1}}$ through coordinatewise multiplication by $w = (w_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A})$, we have the following description of its equations.

PROPOSITION 2. *A point $z \in \triangle^{\mathcal{A}}$ lies in $X_{\mathcal{A},w}$ if and only if*

$$\prod_{\mathbf{a}\in\mathcal{A}} z_{\mathbf{a}}^{\alpha_{\mathbf{a}}} \cdot \prod_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}}^{\beta_{\mathbf{a}}} \;=\; \prod_{\mathbf{a}\in\mathcal{A}} z_{\mathbf{a}}^{\beta_{\mathbf{a}}} \cdot \prod_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}}^{\alpha_{\mathbf{a}}}\,,$$

*for every relation* (5) *among the points of $\mathcal{A}$.*

*Remark* 4. As every component of a point $z \in \triangle^{\mathcal{A}}$ and weight $w$ is nonnegative, we may take arbitrary (positive) roots of the equations in Proposition 2. It follows that we may relax the requirement that the coefficients $\alpha_{\mathbf{a}}$ and $\beta_{\mathbf{a}}$ in (5) are integers and allow them to be any nonnegative numbers such that

$$\sum_{\mathbf{a}\in\mathcal{A}} \alpha_{\mathbf{a}}\mathbf{a} \;=\; \sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}}\mathbf{a} \quad \text{where} \quad \sum_{\mathbf{a}\in\mathcal{A}} \alpha_{\mathbf{a}} \;=\; \sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}} \;=\; 1. \tag{7}$$

That is, $\sum \alpha_{\mathbf{a}}\mathbf{a} = \sum \beta_{\mathbf{a}}\mathbf{a}$ is a point in the convex hull of $\mathcal{A}$ with more than one representation as a convex combination of points of $\mathcal{A}$. Since $\mathcal{A} \subset \mathbb{Z}^d$, we may assume that $\alpha_{\mathbf{a}}, \beta_{\mathbf{a}}$ are rational.

Among all relations (7) are those which arise when two subsets of $\mathcal{A}$ have intersecting convex hulls.

PROPOSITION 3. *Let $\mathcal{F}, \mathcal{G} \subset \mathcal{A}$ be disjoint subsets whose convex hulls meet,*

$$\operatorname{conv}\mathcal{F} \cap \operatorname{conv}\mathcal{G} \;\neq\; \emptyset\,.$$

*Then we have a relation of the form*

$$\sum_{\mathbf{a}\in\mathcal{F}} \alpha_{\mathbf{a}}\mathbf{a} \;=\; \sum_{\mathbf{a}\in\mathcal{G}} \beta_{\mathbf{a}}\mathbf{a} \qquad \textit{where} \qquad \sum_{\mathbf{a}\in\mathcal{F}} \alpha_{\mathbf{a}} \;=\; \sum_{\mathbf{a}\in\mathcal{G}} \beta_{\mathbf{a}} \;=\; 1\,,$$

*with $\alpha_{\mathbf{a}}, \beta_{\mathbf{a}} \geq 0$. Thus*

$$\prod_{\mathbf{a}\in\mathcal{F}} z_{\mathbf{a}}^{\alpha_{\mathbf{a}}} \cdot \prod_{\mathbf{a}\in\mathcal{G}} w_{\mathbf{a}}^{\beta_{\mathbf{a}}} \;=\; \prod_{\mathbf{a}\in\mathcal{G}} z_{\mathbf{a}}^{\beta_{\mathbf{a}}} \cdot \prod_{\mathbf{a}\in\mathcal{F}} w_{\mathbf{a}}^{\alpha_{\mathbf{a}}}\,, \tag{8}$$

*holds on $X_{\mathcal{A},w}$.*

Given a subset $\mathcal{F} \subset \mathcal{A}$, the convex hull of the points $\{e_{\mathbf{f}} \mid \mathbf{f} \in \mathcal{F}\}$ is the simplex $\triangle^{\mathcal{F}}$, which is a face of $\triangle^{\mathcal{A}}$. Under the tautological projection $\pi_{\mathcal{A}}$ of $\triangle^{\mathcal{A}}$ to $\Delta_{\mathcal{A}}$, the simplex $\triangle^{\mathcal{F}}$ maps to the convex hull $\Delta_{\mathcal{F}}$ of $\mathcal{F}$. The *geometric realization* $|\mathcal{S}|$ of a decomposition $\mathcal{S}$ of $\mathcal{A}$ is the union of the simplices $\triangle^{\mathcal{F}}$ for each face $\mathcal{F} \in \mathcal{S}$ of the decomposition $\mathcal{S}$. We call a simplex $\triangle^{\mathcal{F}}$ a *face* of the geometric realization $|\mathcal{S}|$. The images of the faces of the geometric realization $|\mathcal{S}|$ under the tautological projection $\pi_{\mathcal{A}}$ form the faces of the polyhedral decomposition $\mathcal{T}(\mathcal{S})$. Figure 14 illustrates this geometric realization for four regular decompositions of $\mathcal{A} = \{0, 1, 2, 3\}$. For this, $\triangle^{\mathcal{A}}$ is the three-dimensional simplex. For each decomposition $\mathcal{S}$ of $\mathcal{A}$, we show the corresponding polyhedral decomposition of $\Delta_{\mathcal{A}} = [0, 3]$ and its facets.

Suppose that a point $z \in \triangle^{\mathcal{A}}$ lies in the geometric realization $|\mathcal{S}|$ of a decomposition $\mathcal{S}$ of $\mathcal{A}$. Then $z \in \triangle^{\mathcal{F}}$ for some face $\mathcal{F}$ of $\mathcal{S}$, so that its *support* $\{\mathbf{a} \in \mathcal{A} \mid z_{\mathbf{a}} \neq 0\}$ is a subset of $\mathcal{F}$. Conversely, any point $z \in \triangle^{\mathcal{A}}$ whose support is a subset of some face $\mathcal{F}$ of $\mathcal{S}$

lies in $|\mathcal{S}|$. We conclude that $|\mathcal{S}| \subset \triangle^{\mathcal{A}}$ is the vanishing locus of the monomials

$$\{z_{\mathbf{a}} \cdot z_{\mathbf{b}} \mid \{\mathbf{a}, \mathbf{b}\} \not\subset \text{ any face } \mathcal{F} \text{ of } \mathcal{S}\} \cup \{z_{\mathbf{c}} \mid \mathbf{c} \notin \text{ any face } \mathcal{F} \text{ of } \mathcal{S}\}\,. \qquad (9)$$

A point $z \in \triangle^{\mathcal{A}}$ is an *accumulation point* of a sequence $\{X_1, X_2, \dots\}$ of subsets of $\triangle^{\mathcal{A}}$ if, for every $\epsilon > 0$ and every $M$, there is some $m \geq M$ such that distance$(z, X_m) < \epsilon$. Similarly, a point $z$ is an accumulation point of a family $\{X(t) \mid t \in \mathbb{R}_>\}$ if for every $\epsilon > 0$ and $M > 0$, there is some $t > M$ such that distance$(z, X(t)) < \epsilon$, and $z$ is a *limit point* if $\lim_{t\to\infty}$ distance$(z, X(t)) = 0$.

LEMMA 5. *Let $w \in \mathbb{R}^{\mathcal{A}}_{\geq}$ be a weight and $\lambda\colon \mathcal{A} \to \mathbb{R}$ be a lifting function and $w_\lambda(t)$ the corresponding family of weights. Every accumulation point of $\{X_{\mathcal{A},w_\lambda(t)} \mid t \in \mathbb{R}_>\}$ lies in the geometric realization $|\mathcal{S}_\lambda|$.*

PROOF. We will show that a point $y \in \triangle^{\mathcal{A}}$ which does not lie in $|\mathcal{S}_\lambda|$ cannot be an accumulation point of $\{X_{\mathcal{A},w_\lambda(t)}\}$. If $y \in \triangle^{\mathcal{A}}$ but $y \notin |\mathcal{S}_\lambda|$, then by (9) either there are points $\mathbf{a}, \mathbf{b} \in \mathcal{A}$ with $y_{\mathbf{a}} y_{\mathbf{b}} \neq 0$ where $\{\mathbf{a}, \mathbf{b}\}$ do not lie in a common face of $\mathcal{S}_\lambda$, or a single point $\mathbf{c} \in \mathcal{A}$ with $y_{\mathbf{c}} \neq 0$ and $\mathbf{c}$ does not participate in the decomposition $\mathcal{S}_\lambda$. Set $\epsilon := \min\{y_{\mathbf{a}}, y_{\mathbf{b}}\}$ (in the first case) or $\epsilon := y_{\mathbf{c}}$ (in the second case). We will show that if $t$ is sufficiently large and $z \in X_{\mathcal{A},w_\lambda(t)}$, then $\min\{z_{\mathbf{a}}, z_{\mathbf{b}}\} < \epsilon/2$ (in the first case) or $z_{\mathbf{c}} < \epsilon/2$ (in the second case), which will complete the proof.

Suppose that we are in the first case. Then the interior of the segment $\overline{\mathbf{a}, \mathbf{b}}$ meets some face $\Delta_{\mathcal{F}}$ of $\mathcal{T}(\mathcal{S}_\lambda)$. If $\mathcal{F}$ is the minimal such face, then the interiors of $\overline{\mathbf{a}, \mathbf{b}}$ and $\Delta_{\mathcal{F}}$ meet in a point $p$, and so we have the valid relation on $X_{\mathcal{A},w}$,

$$z_{\mathbf{a}}^{\mu} z_{\mathbf{b}}^{\nu} \cdot \prod_{\mathbf{f}\in\mathcal{F}} w_{\mathbf{f}}^{\alpha_{\mathbf{f}}} \;=\; w_{\mathbf{a}}^{\mu} w_{\mathbf{b}}^{\nu} \cdot \prod_{\mathbf{f}\in\mathcal{F}} z_{\mathbf{f}}^{\alpha_{\mathbf{f}}}\,, \qquad (10)$$

by Proposition 3, where

$$p \;:=\; \mu\mathbf{a} + \nu\mathbf{b} \;=\; \sum_{\mathbf{f}\in\mathcal{F}} \alpha_{\mathbf{f}}\mathbf{f} \quad \text{and} \quad \mu + \nu \;=\; 1 \;=\; \sum_{\mathbf{f}\in\mathcal{F}} \alpha_{\mathbf{f}}\,,$$

and the coefficients $\mu, \nu, \alpha_{\mathbf{f}}$ are positive. For $X_{\mathcal{A},w_\lambda(t)}$ the relation (10) becomes

$$z_{\mathbf{a}}^{\mu} z_{\mathbf{b}}^{\nu} \cdot t^{\sum_{\mathbf{f}\in\mathcal{F}} \alpha_{\mathbf{f}}\lambda(\mathbf{f})} \cdot \prod_{\mathbf{f}\in\mathcal{F}} w_{\mathbf{f}}^{\alpha_{\mathbf{f}}} \;=\; \prod_{\mathbf{f}\in\mathcal{F}} z_{\mathbf{f}}^{\alpha_{\mathbf{f}}} \cdot t^{\mu\lambda(\mathbf{a})+\nu\lambda(\mathbf{b})} \cdot w_{\mathbf{a}}^{\mu} w_{\mathbf{b}}^{\nu}\,.$$

Since the lift $\overline{(\mathbf{a}, \lambda(\mathbf{a})), (\mathbf{b}, \lambda(\mathbf{b}))}$ of $\overline{\mathbf{a}, \mathbf{b}}$ does not lie on an upper facet of $P_\lambda$, but the lift of $\Delta_{\mathcal{F}}$ does lie on an upper facet, the point $p$ which is common to $\overline{\mathbf{a}, \mathbf{b}}$ and $\Delta_{\mathcal{F}}$ is lifted lower on the lift of $\overline{\mathbf{a}, \mathbf{b}}$ than on the lift of $\Delta_{\mathcal{F}}$. We thus have the inequality

$$\mu\lambda(\mathbf{a}) + \nu\lambda(\mathbf{b}) \;<\; \sum_{\mathbf{f}\in\mathcal{F}} \alpha_{\mathbf{f}}\lambda(\mathbf{f})\,. \qquad (11)$$

Let $\delta > 0$ be the difference of the two sides of (11). Then points $z \in X_{\mathcal{A},w(t)}$ satisfy

$$z_{\mathbf{a}}^{\mu} z_{\mathbf{b}}^{\nu} \;=\; t^{-\delta} \prod_{\mathbf{f}\in\mathcal{F}} z_{\mathbf{f}}^{\alpha_{\mathbf{f}}} \cdot \frac{w_{\mathbf{a}}^{\mu} w_{\mathbf{b}}^{\nu}}{\prod_{\mathbf{f}\in\mathcal{F}} w_{\mathbf{f}}^{\alpha_{\mathbf{f}}}} \;<\; t^{-\delta} \cdot \frac{w_{\mathbf{a}}^{\mu} w_{\mathbf{b}}^{\nu}}{\prod_{\mathbf{f}\in\mathcal{F}} w_{\mathbf{f}}^{\alpha_{\mathbf{f}}}}\,,$$

as each component of $z \in \triangle^{\mathcal{A}}$ is positive and at most 1.

This inequality implies that if $t$ is sufficiently large, then at least one of the components $z_{\mathbf{a}}, z_{\mathbf{b}}$ is less than $\epsilon/2$, and thus $y$ is not an accumulation point of the sequence. A similar argument in the second case of $\mathbf{c} \notin \mathcal{S}_\lambda$ completes the proof. □

We complete the proof of Theorem 3 by showing that the set of accumulation points of $X_{\mathcal{A},w_\lambda(t)}$ in $\triangle^{\mathcal{F}}$ for $\mathcal{F}$ a facet of $\mathcal{S}_\lambda$ is equal to $X_{\mathcal{F},w|_{\mathcal{F}}}$, as this proves that

$$\lim_{t\to\infty} X_{\mathcal{A},w_\lambda(t)} \;=\; \bigcup_{\mathcal{F}\in\mathcal{S}_\lambda} X_{\mathcal{F},w|_{\mathcal{F}}}\,.$$

LEMMA 6. *Let $\mathcal{F}$ be a face of $\mathcal{S}_\lambda$. Then $X_{\mathcal{F},w|_{\mathcal{F}}}$ is the set of accumulation points of $\{X_{\mathcal{A},w_\lambda(t)} \mid t \in \mathbb{R}_>\}$ that lie in $\triangle^{\mathcal{F}}$, and each point of $X_{\mathcal{F},w|_{\mathcal{F}}}$ is a limit point.*

PROOF. We have that $X_{\mathcal{F},w|_{\mathcal{F}}}$ is the set of points $z \in \triangle^{\mathcal{F}}$ such that

$$\prod_{\mathbf{f}\in\mathcal{F}} z_{\mathbf{f}}^{\alpha_{\mathbf{f}}} \cdot \prod_{\mathbf{f}\in\mathcal{F}} w_{\mathbf{f}}^{\beta_{\mathbf{f}}} \;=\; \prod_{\mathbf{f}\in\mathcal{F}} z_{\mathbf{f}}^{\beta_{\mathbf{f}}} \cdot \prod_{\mathbf{f}\in\mathcal{F}} w_{\mathbf{f}}^{\alpha_{\mathbf{f}}}\,, \qquad (12)$$

whenever $\alpha, \beta \in \mathbb{R}^{\mathcal{F}}_{\geq}$ satisfy

$$\sum_{\mathbf{f}\in\mathcal{F}} \alpha_{\mathbf{f}} \cdot \mathbf{f} \;=\; \sum_{\mathbf{f}\in\mathcal{F}} \beta_{\mathbf{f}} \cdot \mathbf{f} \quad \text{where} \quad \sum_{\mathbf{f}\in\mathcal{F}} \alpha_{\mathbf{f}} \;=\; \sum_{\mathbf{f}\in\mathcal{F}} \beta_{\mathbf{f}} \;=:\; m\,. \qquad (13)$$

The Eq. (12) also holds on $X_{\mathcal{A},w}$, and on $X_{\mathcal{A},w_\lambda(t)}$, it becomes

$$\prod_{\mathbf{f}\in\mathcal{F}} z_{\mathbf{f}}^{\alpha_{\mathbf{f}}} \cdot \prod_{\mathbf{f}\in\mathcal{F}} w_{\mathbf{f}}^{\beta_{\mathbf{f}}} \cdot t^{\sum_{\mathbf{f}} \beta_{\mathbf{f}}\cdot\lambda(\mathbf{f})} \;=\; \prod_{\mathbf{f}\in\mathcal{F}} z_{\mathbf{f}}^{\beta_{\mathbf{f}}} \cdot \prod_{\mathbf{f}\in\mathcal{F}} w_{\mathbf{f}}^{\alpha_{\mathbf{f}}} \cdot t^{\sum_{\mathbf{f}} \alpha_{\mathbf{f}}\cdot\lambda(\mathbf{f})}. \qquad (14)$$

Observe that

$$\frac{1}{m} \sum_{\mathbf{f}\in\mathcal{F}} \alpha_{\mathbf{f}} \cdot \mathbf{f} \;=\; \frac{1}{m} \sum_{\mathbf{f}\in\mathcal{F}} \beta_{\mathbf{f}} \cdot \mathbf{f}$$

is a point in the convex hull of $\mathcal{F}$. Since $\mathcal{F}$ is a face of the decomposition induced by $\lambda$, the function $\lambda$ is affine-linear on $\mathcal{F}$ and so

$$\sum_{\mathbf{f}\in\mathcal{F}} \alpha_{\mathbf{f}} \cdot \lambda(\mathbf{f}) \;=\; \sum_{\mathbf{f}\in\mathcal{F}} \beta_{\mathbf{f}} \cdot \lambda(\mathbf{f})\,.$$

Let this common value be $\delta$. As dividing (14) by $t^{\delta}$ gives (12), we see that (12) is also a valid relation on every member of the family $\{X_{\mathcal{A},w_\lambda(t)} \mid t \in \mathbb{R}_>\}$, whenever $(\alpha_{\mathbf{f}}, \beta_{\mathbf{f}} \mid \mathbf{f} \in \mathcal{F})$ satisfy (13). It follows that this set of Eqs. (12) holds on every accumulation point of the family $\{X_{\mathcal{A},w_\lambda(t)} \mid t \in \mathbb{R}_>\}$, which implies that those accumulation points lying in $\triangle^{\mathcal{F}}$ are a subset of $X_{\mathcal{F},w|_{\mathcal{F}}}$.

To show the other inclusion, for each face $\mathcal{F}$ of $\mathcal{S}_\lambda$, let $X^{\circ}_{\mathcal{F},w|_{\mathcal{F}}}$ consist of those points $z \in X_{\mathcal{F},w|_{\mathcal{F}}}$ with $z_{\mathbf{f}} \neq 0$ for $\mathbf{f} \in \mathcal{F}$. Evidently we have

$$X_{\mathcal{A},w}(\mathcal{S}_\lambda) \;=\; \coprod_{\mathcal{F}\in\mathcal{S}_\lambda} X^{\circ}_{\mathcal{F},w|_{\mathcal{F}}}\,,$$

and so it suffices to prove that every point of $X^{\circ}_{\mathcal{F},w|_{\mathcal{F}}}$ is a limit point of the family $\{X_{\mathcal{A},w_\lambda(t)} \mid t \in \mathbb{R}_>\}$.

Since $\mathcal{F}$ is a face of $\mathcal{S}_\lambda$, there is a vector $\mathbf{v} \in \mathbb{R}^d$ such that the function $\mathcal{A} \to \mathbb{R}$,

$$\mathbf{a} \;\longmapsto\; \mathbf{v} \cdot \mathbf{a} + \lambda(\mathbf{a})$$

is maximized on $\mathcal{F}$ with maximum value $\delta$. That is, if $\mathbf{v}\cdot\mathbf{a} + \lambda(\mathbf{a}) \geq \delta$ with $\mathbf{a} \in \mathcal{A}$, then $\mathbf{a} \in \mathcal{F}$ and $\mathbf{v} \cdot \mathbf{a} + \lambda(\mathbf{a}) = \delta$.

Consider the action of $t \in \mathbb{R}_>$ on $x \in \mathbb{R}^d_>$ where

$$(t * x)_i \;:=\; t^{v_i} x_i\,.$$

Let $z \in X^{\circ}_{\mathcal{F},w|_{\mathcal{F}}}$. Then $z = \varphi_{\mathcal{F},w}(x)$ for some $x \in \mathbb{R}^d_>$, and so

$$\varphi_{\mathcal{A},w}(t * x)_{\mathbf{a}} \;=\; w_{\mathbf{a}} \cdot t^{\mathbf{v}\cdot\mathbf{a}} x^{\mathbf{a}}\,.$$

Thus, under the action $(t.z)_{\mathbf{a}} = t^{\lambda(\mathbf{a})} z_{\mathbf{a}}$ of $\mathbb{R}_>$ on $\mathbb{R}^{\mathcal{A}}_>$, we have

$$t.\varphi_{\mathcal{A},w}(t * x)_{\mathbf{a}} \;=\; w_{\mathbf{a}} \cdot t^{\mathbf{v}\cdot\mathbf{a}+\lambda(\mathbf{a})} x^{\mathbf{a}}\,.$$

Then the line through $t.\varphi_{\mathcal{A},w}(t * x)$ is equal to the line through

$$t^{-\delta}\big(t.\varphi_{\mathcal{A},w}(t * x)\big)\,,$$

whose **a**-coordinate is

$$w_{\mathbf{a}} \cdot t^{\mathbf{v}\cdot\mathbf{a}+\lambda(\mathbf{a})-\delta} x^{\mathbf{a}}\,.$$

Since $\mathbf{v} \cdot \mathbf{a} + \lambda(\mathbf{a}) - \delta \leq 0$ with equality only when $\mathbf{a} \in \mathcal{F}$, we see that the limit of the points $t.\varphi_{\mathcal{A},w}(t * x)$ of $\Delta^{\mathcal{A}}$ as $t \to \infty$ is the point $\varphi_{\mathcal{F},w|_{\mathcal{F}}}(x) = z$, which completes the proof. □

## B. PROOF OF THEOREM 2

Theorem 3 shows that a limit of translates of $X_{\mathcal{A},w}$ by a one-parameter subgroup of $\mathbb{R}^{\mathcal{A}}_>$ (a toric degeneration of $X_{\mathcal{A},w}$) is a regular control surface. This is a special and real-number case of more general results of Kapranov et al. [1991, 1992] concerning all possible toric degenerations of the complexified toric variety $X_{\mathcal{A}}(\mathbb{C})$ that we will use to prove Theorem 2.

Suppose that $\mathcal{A} \subset \mathbb{Z}^d$ is a finite set. We will assume that $\mathcal{A}$ is primitive in that differences of elements of $\mathcal{A}$ span $\mathbb{Z}^d$:

$$\mathbb{Z}^d \;=\; \mathbb{Z}\langle \mathbf{a} - \mathbf{a}' \mid \mathbf{a}, \mathbf{a}' \in \mathcal{A}\rangle\,,$$

that is, $\mathcal{A}$ affinely spans $\mathbb{Z}^d$ (if not, then simply replace $\mathbb{Z}^d$ by the affine span of $\mathcal{A}$). Let $\mathbb{P}^{\mathcal{A}}$ be the complex projective space with homogeneous coordinates $[z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}]$ indexed by elements of $\mathcal{A}$. (These are extensions to all of $\mathbb{P}^{\mathcal{A}}$ of the homogeneous coordinates (3), which were valid for the nonnegative part of $\mathbb{P}^{\mathcal{A}}$.) The complex torus $H := (\mathbb{C}^*)^d$ naturally acts on $\mathbb{P}^{\mathcal{A}}$ with weights given by the set $\mathcal{A}$: $t \in H$ sends the point $z$ with homogeneous coordinates $[z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}]$ to the point $t.z := [t^{\mathbf{a}} z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}]$. Note that $X_{\mathcal{A},\mathbf{1}}(\mathbb{C})$ is the closure of the orbit of $H$ through the point $\mathbf{1} := [1 : \ldots : 1]$. For $w \in (\mathbb{C}^*)^{\mathcal{A}}$, the translate $w.X_{\mathcal{A},\mathbf{1}}(\mathbb{C}) =: X_{\mathcal{A},w}(\mathbb{C})$ is also the closure of the orbit of $H$ through the point $w$ (considered as a point in $\mathbb{P}^{\mathcal{A}}$). Note that $\triangle^{\mathcal{A}}$ is the nonnegative real part [Fulton 1993, Chapter 4] of $\mathbb{P}^{\mathcal{A}}$, and when $w \in \mathbb{R}^{\mathcal{A}}_> \subset (\mathbb{C}^*)^{\mathcal{A}}$, then $X_{\mathcal{A},w}$ is the nonnegative real part of $X_{\mathcal{A},w}(\mathbb{C})$.

A *toric degeneration* of $X_{\mathcal{A},\mathbf{1}}(\mathbb{C})$ is any translate $X_{\mathcal{A},w}(\mathbb{C})$, or any limit of translates

$$\lim_{t\to 0} \lambda(t).X_{\mathcal{A},w}(\mathbb{C}),$$

where $\lambda\colon \mathbb{C}^* \to (\mathbb{C}^*)^{\mathcal{A}}$ is a one-parameter subgroup. This is the same limit as in Section A, its ideal is the limit of the ideals of $\lambda(t).X_{\mathcal{A},w}(\mathbb{C})$ as $t \to 0$. The data of a one-parameter subgroup of $(\mathbb{C}^*)^{\mathcal{A}}$ are equivalent to homomorphisms of abelian groups $\mathbb{Z}^{\mathcal{A}} \to \mathbb{Z}$ and thus to functions $\lambda\colon \mathcal{A} \to \mathbb{Z}$, which explains our notation $\lambda$.

The translates $X_{\mathcal{A},w}(\mathbb{C})$ for $w \in (\mathbb{C}^*)^{\mathcal{A}}$ give a family of subvarieties of $\mathbb{P}^{\mathcal{A}}$, each with the same dimension and degree, and each equipped with an action of $H$. A main result of Kapranov et al. [1991, 1992] identifies all suitable limits of these translates $X_{\mathcal{A},w}(\mathbb{C})$ with the points of a complex projective toric variety $C_{\mathcal{A}}(\mathbb{C})$. The points of $C_{\mathcal{A}}(\mathbb{C})$ in turn are in one-to-one correspondence with all possible complex toric degenerations of $X_{\mathcal{A},w}(\mathbb{C})$ as $w$ ranges over $(\mathbb{C}^*)^{\mathcal{A}}$. For a toric degeneration $X$ of a translate of $X_{\mathcal{A},\mathbf{1}}(\mathbb{C})$, we write $[X]$ for the corresponding point of $C_{\mathcal{A}}(\mathbb{C})$. We will use this result to prove Theorem 2 as follows.

Proof of Theorem 2. Fix control points $\mathcal{B} = \{\mathbf{b}_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^n$ and suppose that $\{w^1, w^2, \ldots\}$ is a sequence of weights in $\mathbb{R}^{\mathcal{A}}_>$ such that the sequence of toric patches $\{Y_{\mathcal{A},w^i,\mathcal{B}} \mid i = 1, 2, \ldots\}$ converges to a set $Y$ in $\mathbb{R}^n$ in the Hausdorff topology.

Consider the corresponding sequence $\{X_{\mathcal{A},w^i}(\mathbb{C}) \mid i = 1, 2, \ldots\}$ of torus translates of $X_{\mathcal{A}}(\mathbb{C})$. This gives a sequence $[X_{\mathcal{A},w^i}(\mathbb{C})]$ of points in the projective toric variety $C_{\mathcal{A}}(\mathbb{C})$. Since $C_{\mathcal{A}}(\mathbb{C})$ is compact, this sequence of points has a convergent subsequence whose limit point is a toric degeneration

$$\lim_{t\to\infty} X_{\mathcal{A},w_{\lambda(t)}}(\mathbb{C}) \;=\; X_{\mathcal{A},w}(\mathcal{S}_\lambda)(\mathbb{C}) \;=\; \bigcup_{\mathcal{F}\in\mathcal{S}_\lambda} X_{\mathcal{F},w|_{\mathcal{F}}}(\mathbb{C})$$

of $X_{\mathcal{A},w}(\mathbb{C})$ for some $w \in (\mathbb{C}^*)^{\mathcal{A}}$ and lifting function $\lambda\colon \mathbb{Z}^{\mathcal{A}} \to \mathbb{Z}$. Replacing the original weights $\{w^i\}$ by this subsequence, we may assume that, as points of $C_{\mathcal{A}}(\mathbb{C})$, we have

$$\begin{aligned}\lim_{i\to\infty}[X_{\mathcal{A},w^i}(\mathbb{C})] \;&=\; [X_{\mathcal{A},w}(\mathcal{S}_\lambda)(\mathbb{C})]\\ &=\; \lim_{t\to\infty}[X_{\mathcal{A},w_{\lambda(t)}}(\mathbb{C})]\,.\end{aligned} \tag{15}$$

The points $[X_{\mathcal{A},w^i}(\mathbb{C})]$ of $C_{\mathcal{A}}(\mathbb{C})$ are translates of the base point $[X_{\mathcal{A}}(\mathbb{C})]$ by elements of $\mathbb{R}^{\mathcal{A}}_> \subset (\mathbb{C}^*)^{\mathcal{A}}$, and so they lie in the nonnegative real part of the toric variety $C_{\mathcal{A}}(\mathbb{C})$, and therefore so does their limit point. But by (15) this limit point is a translate of $X_{\mathcal{A}}(S_\lambda)(\mathbb{C})$, and thus it is a translate by a real weight. This shows that we may take the weight $w$ in (15) to be real.

Theorem 2 will follow from this and the claim that if a sequence $\{[X_i] \mid i \in \mathbb{N}\} \subset C_{\mathcal{A}}(\mathbb{C})$ converges to $[X] \in C_{\mathcal{A}}(\mathbb{C})$ in the analytic topology, then the sequence of subvarieties $\{X_i\}$ converges to $X$ in the Hausdorff metric on subsets of $\mathbb{P}^{\mathcal{A}}$. Given this claim, (15) implies that in the Hausdorff topology on subsets of $\mathbb{P}^{\mathcal{A}}$,

$$\lim_{i\to\infty} X_{\mathcal{A},w^i}(\mathbb{C}) \;=\; \lim_{t\to\infty} X_{\mathcal{A},w_{\lambda(t)}}(\mathbb{C}) \;=\; X_{\mathcal{A},w}(S_\lambda)(\mathbb{C})\,.$$

We may restrict this to their real points to conclude that the limit of patches $X_{\mathcal{A},w^i}$ in $\triangle^{\mathcal{A}}$,

$$\lim_{i\to\infty} X_{\mathcal{A},w^i} \;=\; \lim_{t\to\infty} X_{\mathcal{A},w_{\lambda(t)}} \;=\; X_{\mathcal{A},w}(S_\lambda)\,,$$

is a regular control surface. Since $Y_{\mathcal{A},w^i,\mathcal{B}} = \pi_{\mathcal{B}}(X_{\mathcal{A},w^i})$, the limit $\lim_{i\to\infty} Y_{\mathcal{A},w^i,\mathcal{B}}$ equals

$$\begin{aligned}\lim_{i\to\infty} \pi_{\mathcal{B}}(X_{\mathcal{A},w^i}) \;&=\; \pi_{\mathcal{B}}\big(\lim_{i\to\infty} X_{\mathcal{A},w^i}\big)\\ &=\; \pi_{\mathcal{B}}\big(X_{\mathcal{A},w}(\mathcal{S}_\lambda)\big) \;=\; Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}_\lambda)\,,\end{aligned}$$

which is a regular control surface. This will complete the proof of Theorem 2, once we have proven the claim. □

Proof of claim. As shown in Kapranov et al. [1991, 1992], the projective toric variety $C_{\mathcal{A}}(\mathbb{C})$ is the Chow quotient of $\mathbb{P}^{\mathcal{A}}$ by the group $H = (\mathbb{C}^*)^d$ acting via the weights of $\mathcal{A}$. We explain this construction. Let $D = d! \cdot \mathrm{volume}(\Delta_{\mathcal{A}})$, which is the degree of the projective toric variety $X_{\mathcal{A}}(\mathbb{C})$, as well as any of its translates. Basic algebraic geometry (see Harris [1992, Lecture 21]) gives a complex projective variety $C(D, d, \mathcal{A})$, called the *Chow variety*, whose points are in one-to-one correspondence with $d$-dimensional cycles in $\mathbb{P}^{\mathcal{A}}$ of degree $D$. These are formal linear combinations

$$Z := \sum_{i=1}^{m} D_j Z_j\,, \tag{16}$$

where each coefficient $D_j$ is a nonnegative integer, each $Z_j$ is a reduced and irreducible subvariety of $\mathbb{P}^{\mathcal{A}}$ of dimension $d$, and

$$D = \sum_{j=1}^{m} D_j \cdot \mathrm{degree}(Z_j)\,.$$

In particular all translates $X_{\mathcal{A},w}(\mathbb{C})$ are represented by points of $C(D, d, \mathcal{A})$.

The torus $(\mathbb{C}^*)^{\mathcal{A}}$ acts on $\mathbb{P}^{\mathcal{A}}$ and thus on $C(D, d, \mathcal{A})$, with the points representing the translates $X_{\mathcal{A},w}$ forming a single orbit. The *Chow quotient* $C_{\mathcal{A}}(\mathbb{C})$ is the closure of this orbit in $C(D, d, \mathcal{A})$. An explicit description of $C_{\mathcal{A}}(\mathbb{C})$ may be found in Kapranov et al. [1991, 1992]. We do not need this description to prove Theorem 2, although such a description could be used to help identify the limit control surface whose existence is only asserted by Theorem 2.

The points of $C_{\mathcal{A}}(\mathbb{C})$ correspond to toric degenerations of translates $X_{\mathcal{A},w}(\mathbb{C})$. We describe this, associating a cycle of degree $D$ and dimension $d$ to any toric degeneration. Let $\mathcal{S}_\lambda$ be a regular decomposition of $\mathcal{A}$ induced by a lifting function $\lambda\colon \mathbb{Z}^{\mathcal{A}} \to \mathbb{Z}$. Let $\mathcal{F} \subset \mathcal{A}$ be a facet of $\mathcal{S}_\lambda$. Set $\delta_{\mathcal{F}}$ to be the index in $\mathbb{Z}^d$ of the lattice

$$\mathbb{Z}\langle \mathbf{f} - \mathbf{f}' \mid \mathbf{f}, \mathbf{f}' \in \mathcal{F}\rangle$$

spanned by differences of elements of $\mathcal{F}$. Then $X_{\mathcal{F},w|_{\mathcal{F}}}(\mathbb{C})$ is a subvariety of $\mathbb{P}^{\mathcal{F}}$ of dimension $d$ and degree

$$d! \cdot \mathrm{volume}(\Delta_{\mathcal{F}})/\delta_{\mathcal{F}}\,.$$

The toric degeneration

$$\lim_{t\to\infty} X_{\mathcal{A},w_\lambda(t)}(\mathbb{C}) = \bigcup_{\mathcal{F} \text{ a facet of } \mathcal{S}_\lambda} X_{\mathcal{F},w|_{\mathcal{F}}}(\mathbb{C})$$

(this is a set-theoretic limit) corresponds to the cycle

$$\sum_{\mathcal{F} \text{ a facet of } \mathcal{S}_\lambda} \delta_{\mathcal{F}} X_{\mathcal{F},w|_{\mathcal{F}}}(\mathbb{C})\,,$$

which has degree $D = d! \cdot \mathrm{volume}(\Delta_{\mathcal{A}})$.

We now prove the claim. Following Lawson [1989, Section 2], Kapranov et al. [1991, Section 1] associate to a cycle (16) a current on $\mathbb{P}^{\mathcal{A}}$—the linear functional $\int_Z$ of integrating a smooth $2d$-form over the cycle $Z$. The analytic topology on the Chow variety is equivalent to the weak topology on currents. (The weak topology is the topology of pointwise convergence: A sequence $\{\psi_i \mid i \in \mathbb{N}\}$ of currents converges to a current $\psi$ if and only if for every $2d$-form $\omega$ on $\mathbb{P}^{\mathcal{A}}$ we have $\lim_{i\to\infty} \psi_i(\omega) = \psi(\omega)$, as complex numbers.)

Suppose that $\{[X_i]\} \subset C_{\mathcal{A}}(\mathbb{C})$ converges to $[X]$ in the usual analytic topology on $C_{\mathcal{A}}(\mathbb{C})$,

$$\lim_{i\to\infty} [X_i] = [X]\,.$$

Then the associated currents converge. That is, for every smooth $2d$-form $\omega$, we have

$$\lim_{i\to\infty} \int_{X_i} \omega = \int_X \omega\,. \qquad (17)$$

We use this to show that $\lim_{i\to\infty} X_i = X$, in the Hausdorff metric.

Given a point $x \in X$ and a number $\epsilon > 0$, let $\omega$ be a $2d$-form with $\int_X \omega \neq 0$ which vanishes outside the ball $B(x, \epsilon)$ of radius $\epsilon$ around $x$. Then (17) implies that there is a number $M$ such that if $i > M$, then $\int_{X_i} \omega \neq 0$, and thus $X_i \cap B(x, \epsilon) \neq \emptyset$. Since $X$ is compact, there is some number $M$ such that if $i > M$, then every point of $X$ is within a distance $\epsilon$ of a point of $X_i$.

To complete the proof of the claim, we show that for every number $\epsilon > 0$, there is a number $M$ such that if $i > M$, then every point of $X_i$ lies within a distance $\epsilon$ of $X$. If not, then there is an $\epsilon > 0$ such that for every $M$, there is some $i > M$ such that $X_i$ has a point $x_i$ with $\mathrm{dist}(x_i, X) > \epsilon$. Replacing $\{X_i\}$ by a subsequence, we may assume that each $X_i$ has such a point $x_i$ . It is no loss to assume that the points $x_i$ are smooth. By the compactness of $\mathbb{P}^{\mathcal{A}}$ and of the Grassmannian of $d$-dimensional linear subspaces of $\mathbb{P}^{\mathcal{A}}$, we may replace $\{X_i\}$ by a subsequence and assume that the points $x_i$ converge to a point $x$, and that the tangent spaces $T_{x_i} X_i$ also converge to a linear space $L$. Let $\omega$ be a smooth $2d$-form which vanishes outside of $B(x, \epsilon/2)$ with $\int_L \omega \neq 0$. Then,

$$\lim_{i\to\infty} \int_{X_i} \omega \neq 0\,.$$

But then (17) implies that $\int_X \omega \neq 0$, and so $X \cap B(x, \epsilon/2) \neq \emptyset$, which contradicts our assumption that $X$ is the limit of the $X_i$. □

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